Defining the Jones Polynomial in terms of the Tutte Polynomial

Evan Fang and Jeffrey Chang

Knot Definitions

Knot Definitions

What is a knot?

Knot Definitions

A knot is a loop of string which satisfies the following restrictions:

It is a closed and smooth curve in 3 space

It does not intersect itself anywhere

Knot Definitions

A Knot embedding is a smooth injection $f : \mathbf{S^1} \to \mathbb{R}^3$.

Knot Embedding Equivalency:

We define 2 knot embeddings to be equivalent if they are isotopic.

Knot Definitions

- A knot isotopy is a smooth map $h: \mathbf{S}^1 \times \mathrm{I} \rightarrow \mathbb{R}^3$ that maps each circle to a knot embedding

- The top knot embedding is isotopic to the bottom knot embedding

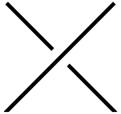
- Knot isotopy is an equivalence relation

- A knot is an equivalence class of knot embeddings

Knot Definitions

Knot Diagram:

Project the knot onto a flat plane

We can use  for crossings, to show over and under.

A knot diagram is alternating if following the strands results in an over under pattern. Every knot has an alternating representation.

However, we need to pick a plane to project it onto. There cannot be a perpendicular to the plane that contains 3 points on the knot. Also, the derivative of the knot cannot be a perpendicular to the plane at any point.

Knot Definitions

Taking 2 different knot embeddings or projection directions can result in different knot diagrams for the same knot.

We need an equivalence relation for knot diagrams expressing when 2 knot diagrams represent the same knot

These are the 3 Reidemeister moves

From Alternating Knots to Graphs

- Shade regions like before
  - Planar Duals
  - Use the one not containing infinity
- Place a vertex in each shaded region
- Place an edge through each crossing

Tutte Polynomial

- Recurrence: T(G) = T(G-e) + T(G/e)
- Base Case
- Need to check:
  - this associates a well-defined polynomial to a graph
  - gives a Knot invariant

Tutte Polynomial

Tutte Polynomial Definition

1. Let $T \in G$ be the subset of all spanning trees in G. Then $\chi_G(x,y) = \sum_T x^r y^s$ where r is the number of externally active edges and y is the number of internally active edges.

2. For all isthmuses e_j we have $\chi_G(x,y) = x \cdot \chi_{G'_j}(x,y)$ and for all edges which are also loops e_k we have $\chi_G(x,y) = y \cdot \chi_{G''_k}(x,y)$. For all other edges e_j which are neither loops nor isthmuses we have $\chi_G(x,y) = \chi_{G'_j}(x,y) + \chi_{G''_j}(x,y)$

Graph Theory Definitions

- Connected Graph
- Cycle
- Tree
- Subgraph
- Spanning Tree
- Random Labeling
- Isthmus

Graph Theory Definitions

- Cyc(T,e)
- Externally Active

Graph Theory Definition

- Cut(T,e)
- Internally Active

Tutte Polynomial Invariant

- Proof
  - Take 2 edges E_1, E_2
  - Only possible change in activity if E_1 is in cyc(T,E_2)
  - Casework

Tutte Polynomial Recurrence

- Proof
  - Assume edge taken is edge maximally labeled
    - Possible by last proof
  - Clearly Satisfies Relationship

Conclusion

- $J(K;t) = f(G;t)*T(G; t, t^{-1})$
  - (Jones Polynomial = weight times Tutte)
  - Weight determined by graph, link in terms of t

Famous Open Questions

- Complex Knots with Jones Polynomial equal to 1?
  - Links already proven
- There is a difficult prove that an (m,n) torus has jones polynomial $t^{(m-1)(n-1)/2}(1-t^{m+1}-t^{n+1}+t^{m+n})/1-t^2$. Is there a simple proof?

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